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Translated by L.K.

PMM U.S.S.R., Vol.50, No.1, pp.75-79, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ON NON-ONE-DIMENSIONAL SELFSIMILAR SOLUTIONS WITH PLANE WAVES IN GAS DYNAMICS*

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A set of new exact selfsimilar solutions is obtained, describing non-one-dimensional adiabatic motions of an ideal gas with plane waves. The solutions show a uniform expansion of the gas in planes perpendicular to the direction of the basic motion. The system of equations of gas dynamics is reduced for these solutions to a system of ordinary differential equations /1/. Problems of a short shock and the propagation of a strong detonation wave in a uniformly expanding gas was solved numerically in /2/, where an exact solution was also found for the problem of a short shock for a special value of the adiabatic index.

1. Let us consider the adiabatic motions of an ideal gas whose parameters are given by the formulas

$$\rho = \frac{a}{x^{k+3}t^{s}} R(\lambda), \quad p = \frac{a}{x^{k+1}t^{s+2}} P(\lambda) \tag{1.1}$$

$$v = \frac{x}{t} V(\lambda), \quad v_i = \varepsilon_i^{(\beta)} \frac{x_i}{t}, \quad \lambda = \frac{x}{bt^{\delta}}$$

Here x_i are rectangular Cartesian coordinates $(x_1 = x)$. The velocity components along the x, x_i axes are denoted by v, v_i, and the index i takes the values 2 and 3 (there is no summation over i). The motion (1.1) is assumed to be either two-dimensional $(\beta = 1,\ \varepsilon_2^{(1)} = 1,\ \varepsilon_3^{(1)} = 0,\ v_2 = x_2/t,\ v_3 = 0)$, or three-dimensional $(\beta = 2,\ \varepsilon_2^{(2)} = 1,\ \varepsilon_3^{(2)} = 1,\ v_2 = x_2/t,\ v_3 = x_3/t)$. The constants a and b are of dimensions $[a] = ML^kT^s$, $[b] = LT^{-\delta}$.

The system of equations of gas dynamics reduces, for such motions, to the following system of ordinary differential equations in the variables $z(\tau) = \gamma P/R$, $V(\tau)$, $R(\tau)$, $\tau = \ln|\lambda|$ /1/:

$$\begin{aligned} dz/dV = {} & z\{z[2 - \varkappa(\gamma - 1) - 2V] + [(\gamma + 1)V - 2 + \beta(\gamma - \\ & 1)](V - \delta)^2 - (\gamma - 1)V(V - 1)(V - \delta)\}(V - \delta)^{-1} \times \\ & [z(\varkappa - \beta - V) + V(V - 1)(V - \delta)]^{-1} \\ V' = {} & [z(\varkappa - \beta - V) + V(V - 1)(V - \delta)][z - (V - \delta)^2]^{-1} \\ R'(V - \delta) = {} & R[s - \beta + (k + 2)V - V'] \\ \varkappa = {} & [s + 2 + \delta(k + 1)]\gamma^{-1} \end{aligned} \tag{1.2}$$

(γ is the adiabatic index $\gamma > 1$).

The last equation of (1.2) can be replaced by the adiabaticity integral /2/

$$PR^{-\gamma} = \mathrm{const}\,[R(V - \delta)]^{\xi}\lambda^{\eta}$$

$$\xi = \frac{2 - (\gamma - 1)s + \delta[k + 1 - \gamma(k + 3)]}{s - \beta + \delta(k + 2)},$$

$$\eta = -\frac{(\gamma + 1)s + 2(k + 2) + \beta[k + 1 - \gamma(k + 3)]}{s - \beta + \delta(k + 2)}$$

The relations on the shocks are written just as in the case of one-dimensional selfsimilar motions /3/

$$R_1(V_1 - \delta) = R_2(V_2 - \delta) \tag{1.3}$$

*Prikl.Matem.Mekhan., 50, 1, 104-109, 1986

$$V_1-\delta+\frac{z_1}{\gamma(V_1-\delta)}=V_2-\delta+\frac{z_2}{\gamma(V_2-\delta)}$$
$$(V_1-\delta)^2+\frac{2z_1}{\gamma-1}=(V_2-\delta)^2+\frac{2z_2}{\gamma-1}$$

Here the indices 1 and 2 denote quantities on different sides of the surface of discontinuity.

2. Let us use as the unperturbed background a uniformly expanding gas in which (ρ_0, p_0 are constants)

$$\rho=\frac{\rho_0}{t^\beta},\quad p=\frac{p_0}{t^{\beta\gamma}},\quad v=0,\quad v_i=\varepsilon_i^{(\beta)}\frac{x_i}{t},\quad i=2,\ 3 \tag{2.1}$$
$$([\rho_0]=ML^{-3}T^\beta,\ [p_0]=ML^{-1}T^{\beta\gamma-2})$$

We shall consider the problem of a gas with parameters (2.1), occupying the right half-space ($x>0$) and escaping into a vacuum. The condition of selfsimilarity requires that the escape of gas into the left-hand half-space begins at the instant $t=0$ corresponding to the singularity in the solution of (2.1). Here ρ_0 and p_0 serve as characteristic dimensional constants; the selfsimilarity index δ and the selfsimilar variable λ are therefore given by the formulas

$$\delta=\frac{2-\beta(\gamma-1)}{2},\quad \lambda=\alpha\sqrt{\frac{\rho_0}{p_0}}\frac{x}{t^\delta}$$

where α is a dimensionless abstract constant.

We further assume that $\gamma<1+2/\beta$. Then the value of δ will lie within the interval $(0,1)$.

Putting $a=\rho_0$ in (1.1), we reduce system (1.2) to the form

$$\frac{dz}{dV}=\frac{z\{2z-[(1+\gamma)V-2\delta](V-\delta)+(\gamma-1)V(V-1)\}}{V[z-(V-1)(V-\delta)]} \tag{2.2}$$
$$V'=-\frac{V[z-(V-1)(V-\delta)]}{z-(V-\delta)^2},\quad R'(V-\delta)=-R(V'+V)$$

The field of integral curves of (2.2) in the (V, z) plane has the following singularities: $O(V=0, z=0)$ is a node, $A(V=0,\ z=\delta^2)$ is a saddle, $B(V=\delta,\ z=0)$ is a saddle, $C(V=1,\ z=0)$ is a node, $D\ (V=2\delta(\gamma+1)^{-1},\ z=\delta(\gamma-1)(\gamma-2\delta+1)(\gamma+1)^{-2})$ is a node, $E(V=0,\ z=\infty)$ is a saddle, $F(V=\infty, z=0)$ is a saddle and $G(V=\infty,\ z=\infty)$ is a degenerate singularity (saddle-node) (Fig.1, the dashed line depicts the parabola $z=(V-\delta)^2$).

The solution of the problem of the escape of a uniformly expanding gas into a vacuum is represented by a sequence of segments of the integral curves OA, AG, GC. The segment OA corresponds to the unperturbed state (2.1), and the singularity A to a weak discontinuity. Matching of the solution at an infinitely distant singularity G can be carried out, for example, by the change of variables $W=(V-\delta)^{-1},\ y=z(V-\delta)^{-2}$. In the (W, y) plane the straight line $W=0$ corresponds to the singularity G, and the separatrix connecting the images of the singularities A and C and intersecting the stright line $W=0$ at some non-singular point, corresponds to the integral curve AGC.

Since the solution has singularities at $t=0$, the boundary with the vacuum recedes instantaneously to infinity ($\lambda=-\infty$), with the singularity C corresponding to it. The asymptotic formulas near this point have the form

$$z=c_1|\lambda|^{-\omega},\ V=1+c_2|\lambda|^{-\chi},\ R=c_3|\lambda|^{1/(\delta-1)}$$

$\omega=2(\beta+1)/\beta,\ \chi=1/(1-\delta)$ when $\gamma\geqslant1+(\beta+1)^{-1}$, $\chi=\omega$ when $\gamma<1+(\beta+1)^{-1}$ (c_1, c_2, c_3 are positive constants.

If $\gamma=1+2(\beta+1)^{-1}$, the problem in question has an exact solution which can be described in the region behind the weak shock by

$$\left|(V-1)^\beta\left(V-\frac{1}{\beta+2}\right)\right|=\frac{1}{\beta+2}\left|\frac{\lambda_*}{\lambda}\right|^{\beta+1} \tag{2.3}$$
$$z=\left(\frac{V-1}{\beta+1}\right)^2,\quad R=\frac{V-1}{(\beta+2)V-1}$$

where $\lambda_*=\alpha\sqrt{(\beta+3)(\beta+1)}$ is the value of the selfsimilar variable corresponding to the weak shock.

In dimensionless variables the solution (2.3) has the form $(-\infty<x<x_*)$

$$\left|\left(v-\frac{x}{t}\right)^\beta\left(v-\frac{1}{\beta+2}\frac{x}{t}\right)\right|=\frac{1}{\beta+2}\left(\frac{x_*}{t}\right)^{\beta+1},\quad v_i=\varepsilon_i^{(\beta)}\frac{x_i}{t}$$
$$\rho=\frac{\rho_0}{t^\beta}\frac{v-x/t}{(\beta+2)v-x/t},\quad p=\frac{p_0}{t^{\beta\gamma}}\frac{t^2}{x_*^2}\frac{(v-x/t)^3}{(\beta+2)v-x/t}$$
$$x_*=\sqrt{(\beta+1)(\beta+3)p_0/\rho_0}\,t^{1/(\beta+1)}$$

The gas parameters in front of the weak shock ($x \geqslant x_*$) are given by the formulas (2.1). The density, pressure and velocity profiles and the pattern of motion for the solution in question for $\beta = 1$, $\gamma = 2$ are shown in Fig.2 (curve *1* corresponds to the distribution of ρ/ρ_*, curve *2* to p/p_*, curve *3* to vt/x_*; and an asterisk denotes the quantities in the unperturbed state (2.1)).

From the conditions on the shocks (1.3) it follows that in the case of $\gamma = 1 + \beta^{-1}$ the singularity $D\,(V = \beta\,(2\beta + 1)^{-1},\ z = {}^1/_2\,(\beta + 1)\,(2\beta + 1)^{-2})$ maps the state of the gas behind the shock wave whose corresponding state in front of the shock is mapped by the point $O\,(V = 0, z = 0)$. Therefore, when $\gamma = 1 + \beta^{-1}$, there exists an exact solution of the selfsimilar problem of the propagation of a strong explosive wave in a uniformly expanding gas. In dimensional variables the solution is represented by the formulas (2.1) (with $p_0 = 0$) for $x > x_*$ where $x_* = b\sqrt{t}$ is the coordinate of the shock wave front (b is a constant), and by the formulas

$$v = \frac{\beta}{2\beta + 1}\,\frac{x}{t}, \quad v_i = \varepsilon_i^{(\beta)}\,\frac{x_i}{t}$$

$$\rho = \frac{\gamma + 1}{\gamma - 1}\,\frac{\rho_0}{t^{\beta}}\left(\frac{x}{x_*}\right)^{2\beta}, \quad p = \frac{\beta\rho_0 b^2}{2\,(2\beta + 1)\,t^{\beta+1}}\left(\frac{x}{x_*}\right)^{2(\beta+1)}$$

with $x < x_*$. The profiles of the gas parameters in this solution and the scheme of motion are shown in Fig.3 (curve *1* is ρ/ρ_*, curve *2* is p/p_*, curve *3* is v/v_*, an asterisk denotes the values directly behind the shock wave front $\beta = 1$, $\gamma = 2$).

3. Let us write in the system of Eqs.(1.2) $\delta = 1$, $\varkappa = \beta$, $k = -1$, $s = \beta\gamma - 2$. Then it simplifies considerably and takes the form

$$z' = z\,[2 - \beta\,(\gamma - 1) - 2V]\,(V - 1)^{-1}, \quad V' = -V \tag{3.1}$$
$$R'\,(V - 1) = R\,[2V + \beta\,(\gamma - 1) - 2]$$

Integrating (3.1) we obtain

$$V = \frac{c_1}{\lambda}, \quad R = \frac{c_3}{z}, \quad z = \begin{cases} c_2\,|V|^{2-\beta(\gamma-1)}\,|V - 1|^{\beta(\gamma-1)}, & c_1 \neq 0 \\ c_2\,|\lambda|^{\beta(\gamma-1)-2}, & c_1 = 0 \end{cases} \tag{3.2}$$

Apart from the solution (3.2) we have, in the case in question, a singular solution of the initial system of equations of gas dynamics, which has the following form in the variables z, V, R, λ :

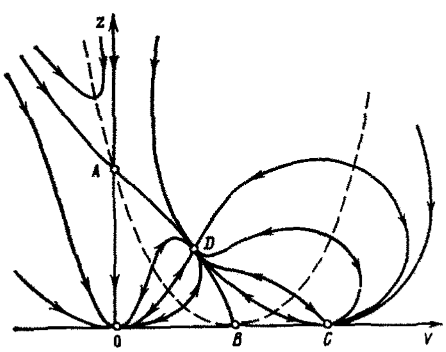

Fig.1

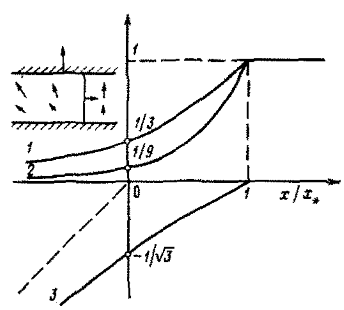

Fig.2

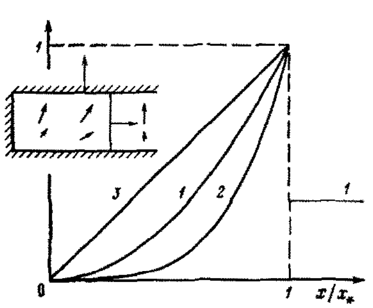

Fig.3

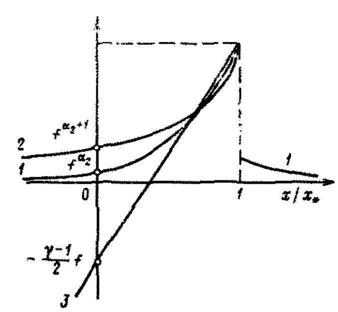

Fig.4

$$z=(V-1)^2,\quad V=\frac{2-\beta(\gamma-1)}{\gamma+1}+\frac{k_1}{\lambda} \tag{3.3}$$
$$R=k_2|V-1|^{\psi}\left|V-\frac{2-\beta(\gamma-1)}{\gamma+1}\right|^{-(\psi+2)},\quad \psi=\frac{2-\beta(\gamma-1)(\gamma+2)}{(\gamma-1)(\beta+1)}$$

where k_1, k_2 are arbitrary constants.

Formulas (3.2) with $c_1=0$ and formulas (3.3) together yield the solution of the problem of the escape, into a vacuum, of a transversely expanding gas with non-uniform density distribution. The solution is written in dimensional variables in the form

$$x_*\leqslant x<\infty:\quad v=0,\quad \rho=\frac{A}{x^{\beta(\gamma-1)}t^{\beta}},\quad p=\frac{p_0}{t^{\beta\gamma}}$$
$$x_V\leqslant x<x_*:\quad v=\frac{2-\beta(\gamma-1)}{\gamma+1}\left[\frac{x}{t}-\left(\frac{\gamma p_0}{A}\right)^{\varphi}\right]$$
$$\rho=\frac{A}{t^{\beta\gamma}}\left(\frac{\gamma p_0}{A}\right)^{-\beta\varphi(\gamma-1)}\left[\frac{2-\beta(\gamma-1)}{\gamma+1}+\frac{(\beta+1)(\gamma-1)}{\gamma+1}\left(\frac{\gamma p_0}{A}\right)^{-\varphi}\frac{x}{t}\right]^{\psi}$$
$$p=\frac{p_0}{t^{\beta\gamma}}\left[\frac{2-\beta(\gamma-1)}{\gamma+1}+\frac{(\beta+1)(\gamma-1)}{\gamma+1}\left(\frac{\gamma p_0}{A}\right)^{-\varphi}\frac{x}{t}\right]^{\psi+2}$$
$$v_i=\varepsilon_i^{(\beta)}\frac{x_i}{t},\quad x_*=\left(\frac{\gamma p_0}{A}\right)^{\varphi}t$$
$$x_V=-\frac{2-\beta(\gamma-1)}{(\beta+1)(\gamma-1)}\left(\frac{\gamma p_0}{A}\right)^{\varphi}t,\quad \varphi=\frac{1}{2-\beta(\gamma-1)}$$

Here x_* and x_V are the coordinates of the weak shock and of the boundary with the vacuum respectively.

We note the existence of an analogue of the Riemann invariant for the solution in question, in the region $x_V\leqslant x\leqslant x_*$ (c is the speed of sound)

$$v-\frac{2-\beta(\gamma-1)}{(\beta+1)(\gamma-1)}c=\text{const}$$

4. Let us consider the selfsimilar problem of a short impact for a uniformly expanding gas whose parameters in the unperturbed motion have the form $(x>0)$

$$v=0,\quad v_i=\varepsilon_i^{(\beta)}\frac{x_i}{t},\quad \rho=\frac{A}{x^{\omega}t^{\beta}},\quad p=0$$

Here A is a constant with dimensions $[A]=ML^{\omega-3}T^{\beta}$, $\beta=0,1,2$ $(\varepsilon_2^{(0)}=\varepsilon_3^{(0)}=0)$, $i=2,3$.

Let $\lambda=x/(bt^{\delta})$ be the selfsimilar variable where b is a certain dimensional constant, and $[b]=LT^{-\delta}$.

We will seek the equation of the trajectory in the (V,z) plane corresponding to the solution of the problem of short impact, in the form

$$z=C(V-1)(V-\delta),\quad C=\text{const} \tag{4.1}$$

The trajectory must pass through the image of the shock wave front $(V=2\delta(\gamma+1)^{-1}$, $z=2\delta^2\gamma(\gamma-1)(\gamma+1)^{-2})$ and through a singularity on the parabola $z=(V-\delta)^2$ with coordinates $(V=\delta(\varkappa-\beta)(\varkappa-\beta+\delta-1)^{-1}$, $z=\delta^2(1-\delta)^2(\varkappa-\beta+\delta-1)^{-2})$, where $\varkappa=[\beta+2+\delta(\omega-2)]\gamma^{-1}$.

Remembering also that the relation (4.1) must satisfy the first equation of (1.2), we obtain the following expressions for the indices δ, ω and the constant C:

$$\delta=\beta+2-(\beta+1)\gamma \tag{4.2}$$
$$\omega=\frac{7+4\beta-(5+4\beta)\gamma}{2\delta},\quad C=\frac{2\gamma\delta}{(2\beta+3)(\gamma-1)}$$

We shall restrict ourselves to considering the case $0\leqslant\omega<1$. Then the possible values of the adiabatic index will be contained within the limits $1<\gamma\leqslant 1+2/(5+4\beta)$. We note that the case $\omega=0$, $\gamma=1+2/(5+4\beta)$ was dealt with in /4, 5/ at $\beta=0$ and in /2/ at $\beta=1,2$. A solution of the problem of a short impact in a medium of variable density $(\beta=0,\ \omega\neq 0)$ was obtained in /6/.

The constant b can be chosen so that condition $\lambda=\lambda_*=1$ holds on the shock wave front. Then the solution of the problem of a short impact is given by the formulas (4.1), (4.2) and the relations

$$V(\lambda)=1-\frac{(2\beta+3)(\gamma-1)}{(\gamma+1)}\frac{1}{\lambda}$$
$$R=\frac{\gamma+1}{\gamma-1}\left|\frac{(\gamma+1)(V-1)}{(2\beta+3)(\gamma-1)}\right|^{\alpha_1}\left|\frac{(\gamma+1)(V-\delta)}{\delta(\gamma-1)}\right|^{\alpha_2}$$
$$\alpha_1=\frac{3+2\beta}{2\delta(\beta+1)},\quad \alpha_2=-\frac{5+4\beta}{2(\beta+1)}$$

In dimensional variables the solution takes the form $(x < x_* = bt^\delta)$

$$v = \frac{x}{t} - \frac{(2\beta+3)(\gamma-1)}{\gamma+1}\frac{x_*}{t}, \quad v_i = \varepsilon_i^{(\beta)}\frac{x_i}{t}$$

$$\rho = \frac{A}{x_*^{\omega}t^{\beta}}\frac{\gamma+1}{\gamma-1}\left\{\frac{\gamma+1}{\delta}\left[\frac{2\beta+3}{\gamma+1} - (\beta+1)\frac{x}{x_*}\right]\right\}^{\alpha_2}$$

$$p = \frac{2\delta(\gamma-1)\rho}{\gamma+1}\left(\frac{x_*}{t}\right)^2\left[\frac{2\beta+3}{\gamma+1} - (\beta+1)\frac{x}{x_*}\right]$$

The corresponding schematic graphs of the density, pressure and velocity distribution are given in Fig.4 (the symbols accompanying the curves are the same as in Fig.3 $f = (2\beta+3)\delta^{-1}$).

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Translated by L.K.

PMM U.S.S.R., Vol.50, No.1, pp.79-85, 1986
Printed in Great Britain
0021-8928/86 $10.00+0.00

INVESTIGATION OF A THREE-DIMENSIONAL HYPERSONIC VISCOUS SHOCK LAYER ON BLUNT BODIES AROUND WHICH FLOW OCCURS AT ANGLES OF ATTACK AND SLIPPAGE*

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The three-dimensional hypersonic flow of dissociating non-equilibrium air past smooth blunt bodies with a catalytic surface is considered. An approximate numerical method of solving the equations of a hypersonic, three-dimensional viscous shock layer (SL for short) is proposed, allowing the study of flows not possessing planes of symmetry. The method is based on introducing, on the body surface, an orthogonal (x^1, x^3), coordinate system attached to the stream lines. The tangents to these stream lines are parallel to the incoming flow velocity vector component lying in the plane parallel to the body surface. The system of equations is written in this coordinate system, and derivatives with respect to the transverse coordinate x^3 of all functions sought are all omitted with exception of the pressure P and the transverse component u^3 of the velocity vector. The derivative $\partial u^3/\partial x^3$ is found from the momentum equation in the x^3 direction differentiated with respect to x^3 and simplified appropriately. The resulting system of equations is identical with the initial system near the critical point and the planes of symmetry, provided that the latter exist. Some results of computing flows at different angles of attack and slippage are given for elliptical paraboloids with various cases of catalytic reactions taking place on the body surface.

Flows past wings of infinite span, for angles of attack and slippage were investigated in /1/. Three-dimensional flows with a plane of symmetry were studied in /2—4/, a triagnular wing at large angles of attack was considered in /2/ and a body of complex shape was considered in /3, 4/.

1. Formulation of the problem.

The equation of a SL can be written, taking the chemical non-equilibrium equations and multicomponent diffusion into account and neglecting

*Prikl.Matem.Mekhan., 50, 1, 110-118, 1986

**Eleonor Arkad'evich Gershbein (1937-1985) was the author of a monograph on hypersonic aerodynamics and of a number of fundamental papers on the theory of boundary and shock layers, gas dynamics and heat transfer in multicomponent gas mixtures.